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REALIZATION THEORY IN HILBERT SPACE

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ABSTRACT

This paper studies the state space representation of time invariant causal, linear input-output operators in continuous time. A realization theorem in the Hilbert space context is established in full generality using unbounded input and output operators. The first step is to construct an abstract state space representation with a prescribed input-output behavior. The second step is to transform this abstract system into a differential equation. The uniqueness problem is discussed in some detail as well as the relation to existing results in realization theory.

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Dietmar Salamon

1. INTRODUCTION

The aim of this paper is to study the state space representation of general infinite dimensional, linear, time invariant, continuous time systems. A time invariant, linear input-output system is described by a linear operator T which associates with each locally square integrable input function on the time interval from 0 to $+\infty$ a corresponding output function which is also locally square integrable and represents the system response to the input excitation. A state space representation is, roughly speaking, a differential equation in some Hilbert space H whose input-output behavior is described by the given operator T. This is known as the realization problem. Its motivation comes from the desire to connect the theory of input-output systems with the control theory for differential equations.

For finite dimensional systems and rational transfer functions the realization problem has been extensively studied and satisfactory solutions can be found e.g. in KALMAN-ARBIB-FALB [12], FUHRMANN [8], [9], KALMAN [11]. The realization of systems with delays, or more generally systems over rings, has been studied by KAMEN [15], [16], SONTAG [23], [24], PANDOLFI [17], ROCHALEAU-SONTAG [20], EISING-HAUTUS [7], SPONG [25] and others using algebraic methods. Distribution theoretic methods have been used in realization theory by KALMAN-HAUTUS [13] and KAMEN [14]. For general infinite dimensional systems the realization problem has been studied by BARAS-BROCKETT [2], BARAS-BROCKETT-FUHRMANN [3], AUBIN-BENSOUSSAN [1], BENSOUSSAN-DELFOUR-MITTER [4], FUHRMANN [10], YAMAMOTO [26] using functional analytic methods. In all of these papers the input operator B and the output operator C in the state space representation are bounded. This leads to smoothing

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requirements on the underlying input-output operator T [26] and also may result in a quite complicated construction of the state space [1], [4].

The essential new feature in the present paper is the consequent use of unbounded input and output operators in the state space representation. Furthermore, the input space U and the output space Y are allowed to be infinite dimensional. This allows us to establish a realization theorem for arbitrary time invariant, causal, linear input-output operators which satisfy a certain exponential bound. The latter condition is necessary in order to construct a realization whose state space is a Hilbert space.

The realization theorem is derived in two steps. The first step is to represent an abstract state space system as a differential equation (section 3). The abstract system satisfies some standard axioms similar to those introduced by KALMAN-ARBIB-FALB [12]. The second step is then to construct such an abstract state space system which has a given input-output behavior (section 4). This construction is done in two different ways and the relation between the resulting two state space models is discussed in detail. In this context the Hankel operator plays a crucial role. The uniqueness problem for the realization is then discussed in connection with various concepts of controllability and observability (section 5). Furthermore, it is shown how the smoothing properties of the Hankel operator are related to the boundedness of the output operator C in the realization (section 6). In a preliminary section we discuss some basic concepts needed for a state space theory of infinite dimensional control systems (section 2).

NOTATION

Let X and Y be Hilbert spaces. Then we denote by $L(X,Y)$ and $L(X) = L(X,X)$ the spaces of bounded linear operators. $L^2[0,T;X]$ denotes the Hilbert space of strongly measurable, square integrable functions from $[0,T]$ into X factorized by the subspace of functions which vanish almost everywhere. By $C[0,T;X]$ we denote the Banach space of continuous functions with the sup norm. $W^{1,2}[0,T;X]$ denotes the Hilbert space of functions $x:[0,T] \to X$ which can be represented in the form

$$x(t) = x(0) + \int_0^t \dot{x}(s)ds , \quad 0 \le t \le T ,$$

for some $\dot{x} \in L^2[0,T;X]$ where the integral is to be understood in the sense of Bochner. $L^2_{\ell oc}[0,\infty;X]$, $C_{\ell oc}[0,\infty,X]$ and $W^{1,2}_{\ell oc}[0,\infty;X]$ denote the Frechét spaces of those functions from $[0,\infty)$ into X whose restrictions to the interval $[0,T]$ are in $L^2[0,T;X]$, $C[0,T;X]$ and $W^{1,2}[0,T;X]$, respectively, for every $T > 0$. The natural dual space of $L^2_{\ell oc}[0,\infty;X]$ is the space $L^2_0[-\infty,0]$ of functions in $L^2[-\infty,0]$ with compact support via the pairing

$$(1.1) \qquad \langle \psi,\phi \rangle = \int_0^\infty \langle \psi(-t),\phi(t) \rangle_X dt$$

for $\psi \in L^2_0[-\infty,0;X]$ and $\phi \in L^2_{\ell oc}[0,\infty;X]$. By $L^2_{0,\ell oc}[\mathbb{R};X]$ we denote the space of all functions in $L^2_{\ell oc}[\mathbb{R};X]$ whose support is bounded to the left. This space is self dual via the pairing

$$(1.2) \qquad \langle \psi,\phi \rangle = \int_{-\infty}^\infty \langle \psi(-t),\phi(t) \rangle_X dt$$

for $\phi,\psi \in L^2_{\ell oc}[\mathbb{R};X]$. For any $\omega \in \mathbb{R}$ the Hilbert spaces

$$L^2_\omega[0,\infty;X] = \{\phi \in L^2_{\ell oc}[0,\infty;X] \mid \int_0^\infty e^{-2\omega t}\|\phi(t)\|^2_X dt < \infty\},$$

$$L^2_\omega[-\infty,0;X] = \{\psi \in L^2_{\ell oc}[-\infty,0;X] \mid \int_0^\infty e^{2\omega t}\|\psi(-t)\|^2_X dt < \infty\}$$

are dual to each other via the pairing (1.1). We also introduce the spaces

$$W^{1,2}_\omega[0,\infty;X] = \{\phi \in W^{1,2}_{\ell oc}[0,\infty;X] \mid \phi,\dot{\phi} \in L^2_\omega[0,\infty;X]\} ,$$

$$W^{1,2}_\omega[-\infty,0;X] = \{\psi \in W^{1,2}_{\ell oc}[-\infty,0;X] \mid \psi,\dot{\psi} \in L^2_\omega[-\infty,0;X]\} .$$

For any interval $I \subset \mathbb{R}$ the shift operator τ_t on $L^2_{\ell oc}[I;X]$ is defined by

$$(\tau_t\phi)(s) = \begin{cases} \phi(t+s), & t+s \in I, \\ 0 \quad , & t+s \notin I, \end{cases}$$

for $t \in \mathbb{R}$, $s \in I$, $\phi \in L^2_{\ell oc}[I;X]$. The symbol χ_I stands for the characteristic function of the interval I.

2. STATE SPACE THEORY

In this section we give a brief overview over the basic concepts in a state space approach for time invariant, linear, infinite dimensional control systems. In order to prepare the grounds for a realization theorem in full generality we will have to deal with unbounded input and output operators. More precisely, we consider the control system described by the differential equations

$$(2.1)\qquad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t)\ , \quad x(0) = x_0\ , \\ y(t) &= C(\mu I - A)^{-1}\big(\mu x(t) - \dot{x}(t)\big) + T_\mu u(t)\ , \end{aligned}$$

with input $u(t) \in U$, output $y(t) \in Y$ and state $x(t) \in H$. All three spaces U, H, Y are Hilbert spaces and we assume that A is the infinitesimal generator of a strongly continuous semigroup $S(t) \in \mathcal{L}(H)$. Furthermore, we consider $W = \mathcal{D}(A) \subset H$ and $V^* = \mathcal{D}(A^*) \subset H^*$ as Hilbert spaces with the respective graph norms so that

$$W \subset H \subset V$$

with continuous, dense injections. Of course, $S(t)$ is a strongly continuous semigroup on all three spaces W, H, V and A can be considered as a bounded operator $A \in \mathcal{L}(W,H) \cap \mathcal{L}(H,V)$. Finally, we assume that the opeators $B \in \mathcal{L}(U,V)$, $C \in \mathcal{L}(W,Y)$ and $T_\mu \in \mathcal{L}(U,Y)$, $\mu \notin \sigma(A)$, satisfy the compatibility condition.

$$(2.2)\qquad T_\mu - T_\lambda = (\lambda - \mu)C(\mu I - A)^{-1}(\lambda I - A)^{-1}B$$

for $\lambda,\mu \notin \sigma(A)$. This condition guarantees that the expression for the output in (2.1) is independent of the choice of $\mu \notin \sigma(A)$. We also point out that the output equation in (2.1) only makes sense if the solution

$$(2.3)\qquad x(t;x_0,u) = S(t)x_0 + \int_0^t S(t-s)Bu(S)ds,\ t > 0\ ,$$

of (2.1) is continuously differentiable in H. If that is the case then we denote the output of (2.1) by

$$(2.4)\qquad y(t;x_0,u) = C(\mu I-A)^{-1}\big(\mu x(t;x_0,u) - Ax(t;x_0u) - Bu(t)\big) + T_\mu u(t), \quad t \geq 0 .$$

In particular, equation (2.4) makes sense whenever $u(\cdot) \in W^{2,2}[0,T;U]$ and $Ax_0 + Bu(0) \in H$.

DEFINITION 2.1

The semigroup control system (2.1) is said to be well posed if for every $T > 0$ there exists a constant $c_T > 0$ such that the inequalities

$$\|x(t;x_0,u)\|_H^2 \leq c_T^2 \left[\|x_0\|_H^2 + \int_0^t \|u(s)\|_U^2 ds\right] ,$$

$$\int_0^T \|y(s;x_0u)\|_Y^2 ds \leq c_T^2 \left[\|x_0\|_H^2 + \int_0^T \|u(s)\|_U^2 ds\right] ,$$

hold for all $x_0 \in H$, $u(\cdot) \in W^{2,2}[0,T;U]$ with $Ax_0 + Bu(0) \in H$ and all $t \in [0,T]$.

For a wellposed system (2.1) we introduce the associated input/output operator $\mathcal{T} : L^2_{\ell oc}[0,\infty,U] \to L^2_{\ell oc}[0,\infty;Y]$ which is defined by

$$(2.5)\qquad \mathcal{T}u = y(\cdot;0,u) .$$

This is possible by continuous extension of the expression $y(t;x_0,u)$ to arbitrary $x_0 \in H$ and $u \in L^2_{\ell oc}[0,\infty;U]$ using the inequality in Definition 2.1. We also introduce the input-state map $\mathcal{B} : L^2_0[-\infty,0;U] \to H$ and the state-output map $\mathcal{C} : H \to L^2_{\ell oc}[0,\infty;Y]$ defined by

$$(2.6)\qquad \mathcal{B}u = \int_0^\infty S(t)Bu(-t)dt , \quad u \in L^2_0[-\infty,0;U] ,$$

$$(2.7)\qquad \mathcal{C}x_0(t) = CS(t)x_0 , \quad t \geq 0 , \quad x_0 \in W .$$

The composition $\mathcal{H} = \mathcal{C}\mathcal{B}$ of these operators is usually called the Hankel operator of the control system (2.1).

$$\begin{tikzcd} L^2_0[-\infty,0;U] \arrow[rr, "\mathcal{H}"] \arrow[dr, "\mathcal{B}"'] & & L^2_{\ell oc}[0,\infty;Y] \\ & H \arrow[ur, "\mathcal{C}"'] & \end{tikzcd}$$

At the end of the section we collect a few fundamental properties of wellposed semigroup control systems.

LEMMA 2.2

Suppose that the system (2.1) is wellposed and let $\omega > \omega_0 = \lim\limits_{t\to\infty} t^{-1}\log\|S(t)\|$. Then the following statements hold.

(i) If $x_0 \in H$ and $u \in L^2[0,T;U]$ then $x(\cdot) = x(\cdot;x_0 u) \in C[0,T;H]$ $W^{1,2}[0,T;V]$ satisfies equation (2.1) in the space V for almost every $t \in [0,T]$.

(ii) If $u \in W^{1,2}[0,T;U]$ and $Ax_0 + Bu(0) \in H$, then $x(\cdot;x_0,u) \in C^1[0,T;H]$, $y(\cdot;x_0,u) \in W^{1,2}[0,T;Y]$ and $\dot{x}(t;x_0,u) = x(t;Ax_0+Bu(0),\dot{u})$, $\dot{y}(t;x_0,u) = y(t;Ax_0+Bu(0),\dot{u})$ for (almost) every $t \in [0,T]$.

(iii) There exists a constant $c > 0$ such that the inequalities

$$\int_0^\infty e^{-2\omega t}\|CS(t)x_0\|_Y^2 dt \le c^2\|x_0\|_H^2$$

$$\left\|\int_0^T S(t)Bu(-t)dt\right\|_H^2 \le c^2 \int_0^T e^{2\omega t}\|u(-t)\|_U^2 dt$$

hold for every $x_0 \in W$ every $T > 0$ and every $u \in W^{1,2}[-T,0;U]$ with $u(-T) = 0$.

PROOF: The statements (i) and (ii) have been proved in [22, Lemma 2.5]. In order to establish statement (iii) let us first note that $\|S(T)\| \le e^{\omega T}$ for $T > 0$ sufficiently large. Hence we obtain from the inequality in Definition 2.1 that

$$\int_0^\infty e^{-2\omega t}\|CS(t)x_0\|^2 dt$$

$$= \sum_{k=0}^\infty \int_0^T e^{-2\omega t} e^{-2\omega kT}\|CS(t)S(kT)x_0\|_Y^2 dt$$

$$\leq \sum_{k=0}^\infty e^{-2\omega kT}\int_0^T \|CS(t)S(kT)x_0\|_Y^2 dt$$

$$\leq c_T \sum_{k=0}^\infty e^{-2\omega kT}\|S(T)\|^{2k}\|x_0\|_H^2$$

$$= c_T e^{2\omega T}\left(e^{2\omega T}-\|S(T)\|^2\right)^{-1}\|x_0\|_H^2$$

for $x_0 \in W$. The second assertion in statement (iii) follows by duality. $\square$

Statement (iii) of the previous Lemma says that the range of $\mathcal{C}$ lies in $L^2_\omega[0,\infty;Y]$ and that $\mathcal{B}$ extends to a bounded operator on $L^2_\omega[-\infty,0;U]$. Therefore we obtain the following commuting diagram

$$\begin{tikzcd} L^2_\omega[-\infty,0;U] \arrow[rr, "H"] \arrow[dr, "\mathcal{B}"'] & & L^2_\omega[0,\infty;Y] \\ & H \arrow[ur, "\mathcal{C}"'] & \end{tikzcd}$$

<u>LEMMA 2.3</u>

<u>(i)</u> <u>If</u> $x_0 \in W$ <u>then</u> $\phi = \mathcal{C}x_0 \in W^{1,2}_\omega[0,\infty;Y]$ and $\dot{\phi} = \mathcal{C}Ax_0$.

<u>(ii)</u> <u>If</u> $\psi \in W^{1,2}_\omega[-\infty,0;U]$ <u>with</u> $\psi(0) = 0$, <u>then</u> $\psi \in W$ <u>and</u> $A\mathcal{B}\psi = \dot{\psi}$.

<u>PROOF</u>: Statement (i) follows immediately from Lemma 2.2. If $\psi \in W^{1,2}_\omega[-\infty,0;U]$ is supported on $[-T,0]$ and satisfies $\psi(0) = 0$ then it follows also from Lemma 2.2 with $u(t) = \psi(t-T)$, $t \geq 0$, that $\mathcal{B}\psi = x(T;0,u) \in W$ and $A\mathcal{B}\psi = \dot{x}(T;0,u) = x(T;0,\dot{u}) = \mathcal{B}\dot{\psi}$. In

general, statement (ii) follows from the fact that the functions in $W_{\omega}^{1,2}[-\infty,0;U]$ with compact support are dense and that A is a closed operator on H. $\square$

A more detailed discussion of semigroup control systems of the form (2.1) can be found in SALAMON [22]. Furthermore, it is shown in [22] how large classes of partial and functional differential equations can be represented in the framework of equation (2.1). For the fundamental properties of strongly continuous semigroups the reader is referred to PAZY [18]. The Hankel operator plays a central role in the extensive studies by FUHRMANN [10] on discrete and continuous time systems in Hilbert space with bounded input and output operators. For systems with unbounded input and output operators the Hankel operator has recently been studied by CURTAIN [5].

3. A REPRESENTATION THEOREM

A <u>time invariant, linear control system</u> consists of three Hilbert spaces U, H, Y and two continuous, linear maps

$$\begin{cases} H \times L^2_{\ell oc}[0,\infty;U] \longrightarrow {}_{\ell oc}[0,\infty;H] \\ (x_0,u) \longrightarrow x(\cdot;x_0,u) \end{cases}$$

$$\begin{cases} H \times L^2_{\ell oc}[0,\infty;U] \longrightarrow L^2_{\ell oc}[0,\infty;Y] \\ (x_0,u) \longrightarrow y(\cdot;x_0,u) \end{cases}$$

which satisfy the following conditions.

1. CAUSALITY

   $u(t) = 0 \ \forall\, t < T \implies x(t;0,u) = 0, \ \ y(t;0,u) = 0 \quad \forall\, t < T$.

2. INITIAL CONDITION

   $x(0;x_0,u) = x_0 \quad \forall\, x_0 \in H \quad \forall\, u \in L^2_{\ell oc}[0,\infty;U]$.

3. TIME INVARIANCE

   $x(t+s;x_0,u) = x(t;x(s;x_0,u),\tau_s u)$,

   $y(t+s;x_0,u) = y(t;x(s;x_0,u),\tau_s u)$,

   $\forall\, x_0 \in H, \quad \forall\, u \in L^2_{\ell oc}[0,\infty;U] \quad \forall\, t,s > 0$.

This definition is similar to the one given by KALMAN-ARBIB-FALB [12] and every wellposed semigroup control system of the form (2.1) satisfies its requirements. The converse statement is formulated in the following Theorem.

<u>THEOREM 3.1</u>

<u>Every time invariant, linear control system can be represented in a unique way by operators</u> A, B, C, T_μ <u>via</u> (2.3) <u>and</u> (2.4).

The remainder of this section is devoted to the proof of this result. The main tool is the next Lemma.

$\mathcal{Y} = L^2_{\omega+\varepsilon}[0,\infty;Y]$ for any $\varepsilon > 0$. The only possibility that this concept is independent of the choice of the space $\mathcal{Y}$ is that the system Σ is continuously observable in finite time. This is however a much more restrictive assumption. It means that the operator which maps $x_0 \in H$ into the output segment $y(\cdot;x_0,0) \in L^2[0,T;Y]$ is injective and has a closed range for some $T > 0$. The existence of a realization of $\mathcal{T}$ with this property requires that there exist constants $T > 0$, $M > 1$, $\omega > 0$ such that the following inequality holds for all $t > 0$ and all $\psi \in L_0^2[-\infty,0;U]$

$$(5.2) \qquad \int_t^{t+T} \| \mathcal{H}\psi(s)\|_Y^2 ds \leq Me^{\omega t}\int_0^T \| \mathcal{H}\psi(s)\|_Y^2 ds .$$

This condition is also sufficient (see YAMAMOTO [26, Theorem 7.4] for the single input-single output case).

EXAMPLE 5.1

Let $\alpha_n > 0$ be a summable sequence such that

$$(5.3) \qquad \operatorname{rank} \begin{bmatrix} \alpha_0 & \cdots & \alpha_n \\ \vdots & & \vdots \\ \alpha_n & \cdots & \alpha_{2n} \end{bmatrix} = n + 1 \quad \forall\, n \in \mathbb{N} .$$

Then the input-output operator $\mathcal{T}$ on $L^2_{0,\ell oc}[\mathbb{R}]$ defined by

$$(5.4) \qquad u(t) = \sum_{n=0}^{\infty} \alpha_n u(t-n)$$

is ω-stable with $\omega = 0$. Furthermore, for any $T > 0$ and any $\phi \in L^2[0,T]$ there exists a (unique) $\psi \in L^2[-T,0]$ such that $\mathcal{H}\psi(t) = \phi(t)$ for $0 \leq t \leq T$. This shows that there is no inequality of the form (5.2) for the Hankel operator $\mathcal{H}$ associated with (5.4). Therefore the realization constructed by YAMAMOTO [26] does not have a Hilbert space as a state space in contrast to our result (Theorem 4.3). This is a consequence of his concept of continuous observability in the output space $\mathcal{Y} = L^2_{\ell oc}[0,\infty]$. In other words the state space of the realization is chosen to be the closure of range $\mathcal{H}$ in $L^2_{\ell oc}[0,\infty]$ rather than $L^2[0,\infty]$ and is therefore not a Hilbert space. The construction in [26] leads to a Hilbert space if and only if condition (5.2) is satisfied.

This operator will not be an isomorphism, in general, unless the original Hankel operator $\mathcal{H} : H_U \longrightarrow H_Y$ has a closed range. In fact, it has been observed by BARAS-BROCKETT-FUHRMANN [2] that two canonical realizations need not be isomorphic. One way out would be to introduce a stronger notion of observability or reachability. This approach has been proposed by YAMAMOTO [26] and can in our context be formulated as follows.

Let the control system $\Sigma = (A,B,C,T_\mu)$ be given and let the operators $\mathcal{B} : L_0^2[-\infty,0;U] \to H$, $\mathcal{C} : H \to L_{\ell oc}^2[0,\infty;Y]$ be defined by (2.6) and (2.7) respectively. Furthermore, let $\mathcal{U} \supset L_0^2[-\infty,0;U]$ and $\mathcal{Y} \subset L_{\ell oc}^2[0,\infty;Y]$ be complete topological vectorspaces such that $\mathcal{B}$ extends to a continuous linear operator from $\mathcal{U}$ into H and $\mathcal{C}$ maps H continuously into $\mathcal{Y}$. Then Σ is said to be <u>exactly $\mathcal{U}$ - reachable</u> if $\mathcal{B}\mathcal{U} = H$. It is said to be <u>continuously $\mathcal{Y}$ - observable</u> if it is observable and $\mathcal{C}$ has a closed range in $\mathcal{Y}$. The latter concept has been used by YAMAMOTO [26] with $\mathcal{Y} = L_{\ell oc}^2[0,\beta;Y]$.

Note that the system Σ_Y is in fact continuously $\mathcal{Y}$ - observable with $\mathcal{Y} = L_\omega^2[0,\infty;Y]$. Likewise, the system Σ_U is exactly $\mathcal{U}$ - reachable with $\mathcal{U} = L_\omega^2[-\infty,0;U]$. Therefore the reduced system $\bar{\Sigma}_Y$ is reachable and continuously $\mathcal{Y}$ - observable.

Now suppose that $\Sigma = (A,B,C,T_\mu)$ is any other realization of $\mathcal{T}$ which is reachable and continuously $\mathcal{Y}$ - observable. Then the associated operator $\mathcal{C} : H \to \mathcal{Y} = H_Y$ has a closed range and $\mathcal{B} : L_0^2[-\infty,0;U] \to H$ has a dense range. By (5.1), this implies

$$\bar{H}_Y = \mathrm{cl}(\mathrm{range}\ \mathcal{H}) = \mathrm{cl}(\mathrm{range}\ \mathcal{C}\mathcal{B}) = \mathrm{range}\ \mathcal{C}.$$

Therefore $\mathcal{C} : H \to \bar{H}_Y$ provides an isomorphism between the systems Σ and $\bar{\Sigma}_Y$ (compare diagram (4.10)). In the same spirit one can prove the existence and uniqueness for realizations which are either observable and exactly $\mathcal{U}$ - reachable or reachable and continuously $\mathcal{Y}$ - observable for any suitable space $\mathcal{U}$ or $\mathcal{Y}$.

In my opinion, however, the concepts of continuous $\mathcal{Y}$ - observability and exact $\mathcal{U}$ - reachability are much too strong for infinite dimensional system. For example, the control system Σ_Y of Theorem 4.3 will in general not be continuously $\mathcal{Y}$ - observable with

5. OBSERVABILITY AND REACHABILITY

A semigroup control system $\Sigma = (A,B,C,T_\mu)$ of the form (2.1) is said to be observable if the unobservable subspace

$$N = \{x_0 \in H \mid y(t;x_0;0) = 0 \ \forall\, t > 0\}$$

is zero. It is said to be reachable if the reachable subspace

$$R = \{x(t;0,u) \mid t > 0,\ u \in L^2_{\ell oc}[0,\infty;U]\}$$

is dense in H. Any semigroup control system can be made canonical that is reachable and observable by restricting the state space to $\mathrm{cl}(R) \subset H$ and then factorizing it through the subspace $\mathrm{cl}(R) \cap N$. This procedure does not change the input-output operator T which corresponds to Σ.

Note that the semigroup control system Σ_Y of Theorem 4.3 is already observable and its reachable subspace is

$$R_Y = \{ H\psi \mid \psi \in L^2_0[-\infty,0;U] \} \subset H_Y$$

(Compare formula (4.2)). Likewise, the semigroup control system Σ_U of Theorem 4.4 is reachable and its unobservable subspace is

$$N_U = \{\psi \in H_U \mid H\psi = 0\} \subset H_U$$

(compare formula (4.5)). Applying the above procedure to these systems we obtain two canonical realizations $\bar{\Sigma}_U$ and $\bar{\Sigma}_Y$ in the Hilbert spaces

$$\bar{H}_U = H_U/\ker H\ , \quad \bar{H}_Y = \mathrm{cl}(\mathrm{range}\ H). \tag{5.1}$$

These are related by the (reduced) Hankel operator

$$\bar{H} : \bar{H}_U \longrightarrow \bar{H}_Y\ .$$

MANITIUS [6], SALAMON [21], PRITCHARD-SALAMON [19]). Here the output plays the role of the forcing function and the input the role of the initial function.

Now suppose that the input-output operator $\mathcal{T}$ is realized by a third semigroup control system $\Sigma = (A,B,C,T_\mu)$ where A generates a semigroup $S(t) \in \mathcal{L}(H)$ of exponential type $\omega_0 = \lim_{t\to\infty} t^{-1}\log\|S(t)\| < \omega$. Then the Hankel operator $\mathcal{H}$ can be decomposed in the form

$$\begin{tikzcd}
H_U \arrow[rr, "\mathcal{H}"] \arrow[dr, "\mathcal{B}"'] & & H_Y \\
& H \arrow[ur, "\mathcal{C}"'] &
\end{tikzcd}$$

where the operators

$$\mathcal{B} \in \mathcal{L}(H_U,H) \cap \mathcal{L}(W_U,W) \cap \mathcal{L}(V_U,V)$$

$$\mathcal{C} \in \mathcal{L}(H,H_Y) \cap \mathcal{L}(W,W_Y) \cap \mathcal{L}(V,V_Y)$$

are defined by (2.6) and (2.7) (see Lemma 2.3). These operators make the following diagram commute.

$$(4.10)\qquad \begin{tikzcd}
& V_U \arrow[r, phantom, "\supset"] \arrow[d] \arrow[dr, phantom, "\mathcal{B}"] & H_U \arrow[r, "S_U(t)"] \arrow[d] & H_U \arrow[r, phantom, "\supset"] \arrow[d] \arrow[dr, phantom, "\mathcal{B}"] & W_U \arrow[d] \arrow[dr, "C_U"] & \\
U \arrow[ur, "B_U"] \arrow[r, "B"] \arrow[dr, "B_Y"'] & V \arrow[r, phantom, "\supset"] \arrow[d] \arrow[dr, phantom, "\mathcal{C}"] & H \arrow[r, "S(t)"] \arrow[d] & H \arrow[r, phantom, "\supset"] \arrow[d] \arrow[dr, phantom, "\mathcal{C}"] & W \arrow[r, "C"] \arrow[d] & Y \\
& V_Y \arrow[r, phantom, "\supset"] & H_Y \arrow[r, "S_Y(t)"] & H_Y \arrow[r, phantom, "\supset"] & W_Y \arrow[ur, "C_Y"'] &
\end{tikzcd}$$

Likewise, the control system (4.4), (4.5) is represented by the following spaces and operators

$$
(4.7)\qquad
\left\{
\begin{array}{ll}
H_U = L^2_\phi[-\infty,0;U] \ , & H^*_U = L^2_\omega[0,\infty;U] \ , \\
W_U = \{\psi \in W^{1,2}_\omega[-\infty,0;U] \,|\, \psi(0) = 0\} \ , & V^*_U = W^{1,2}_\omega[0,\infty;U] \ , \\
A_U\psi = \dot{\psi} \ , & A^*_U\phi = \dot{\phi} \ , \\
S_U(t)\psi = \tau_t\psi \ , & S^*_U(t)\phi = \tau_t\phi \ , \\
C_U\psi = \mathcal{H}\psi(0) \ , & B^*_U\phi = \phi(0) \ ,
\end{array}
\right.
$$

for $\psi \in W_U$ and $\phi \in V^*_U$. This system will be denoted by Σ_U.

Both systems have the same transfer operator T_μ which may be defined in terms of the Laplace transform, that is

$$
(4.8)\qquad T_\mu u_0 = \mu(1-e^{-\mu T})^{-1} \int_0^\infty e^{-\mu t} y(t;u_0\chi_{[0,T]})dt
$$

for $u_0 \in U$, $T > 0$ and $\mathrm{Re}\mu$ sufficiently large.

If the input-output operator $\mathcal{T}$ *is ω-stable then the Hankel operator*

$$
\mathcal{H} \in \mathcal{L}(H_U,H_Y) \cap \mathcal{L}(W_U,W_Y) \cap \mathcal{L}(V_U,V_Y)
$$

(Lemma 4.2) provides a natural homomorphism between the systems Σ_U and Σ_Y. More precisely, the following diagram commutes

$$
(4.9)\qquad
\begin{tikzcd}
 & V_U \arrow[dd, "\mathcal{H}"] & \supset & H_U \arrow[rr, "S_U(t)"] \arrow[dd] & & H_U \arrow[dd] & \supset & W_U \arrow[dd, "\mathcal{H}"'] \arrow[dr, "C_U"] & \\
U \arrow[ur, "B_U"] \arrow[dr, "B_Y"'] & & & & & & & & Y \\
 & V_Y & \supset & H_Y \arrow[rr, "S_Y(t)"] & & H_Y & \supset & W_Y \arrow[ur, "C_Y"'] &
\end{tikzcd}
$$

Note that the role of the Hankel operator in this context is very similar to the role of the structural operator F in the theory of functional differential equations (DELFOUR-

PROOF: As our state space we choose $H = L^2_\omega[-\infty,0;U]$ and define

$$(4.4)\qquad x(t;\psi,u)(s) = \begin{cases}\psi(t+s), & s < -t\ ,\\ u(t+s), & -t < s < 0\ ,\end{cases}$$

$$(4.5)\qquad y(t;\psi,u) = \mathcal{H}\psi(t) + y(t;u)\ ,$$

for $\psi \in H$, $u \in L^2_{loc}[0,\infty;U]$, $t < 0$. These maps define a time invariant linear control system in the sense of section 3 and hence the statement follows from Theorem 3.1. □

Note that both in Theorem 4.3 and in Theorem 4.4 the constructed semigroup operator $S(t)$ has exactly the norm $e^{\omega t}$. This allows the following conclusion.

COROLLARY 4.5

Let $\mathcal{T}$ be a time invariant, causal, linear input-output operator and let $\omega_0 \in R$ be given. Then the following statements are equivalent.

(i) $\mathcal{T}$ is ω-output-stable for all $\omega > \omega_0$.

(ii) $\mathcal{T}$ is ω-input-stable for all $\omega > \omega_0$.

(iii) $\mathcal{T}$ is ω-stable for all $\omega > \omega_0$.

We obtain from Theorem 3.1 that the control system (4.2), (4.3) is represented by the following spaces and operators

$$(4.6)\qquad \begin{cases} H_Y = L^2_\omega[0,\infty;Y]\ , & H^*_Y = L^2_\omega[-\infty,0;Y]\ ,\\ W_Y = W^{1,2}_\omega[0,\infty;Y]\ , & V^*_Y = \{\psi \in W^{1,2}_\omega[-\infty,0;Y]\,|\,\psi(0) = 0\}\ ,\\ A_Y\phi = \dot\phi\ , & A^*_Y\psi = \dot\psi\ ,\\ S_Y(t)\phi = \tau_t\phi\ , & S^*_Y(t)\psi = \tau_t\psi\ ,\\ C_Y\phi = \phi(0)\ , & B^*_Y\psi = \mathcal{H}^*\psi(0)\ ,\end{cases}$$

for $\phi \in W_Y$ and $\psi \in V^*_Y$. This system will be denoted by Σ_Y.

$$
\begin{aligned}
&x(t+s;\phi,u)\\
&= \tau_{t+s}\phi + \tau_{t+s}y(\cdot;u\cdot\chi_{[0,t+s]})\\
&= \tau_t\big(\tau_s\phi + \tau_s y(\cdot;u\cdot\chi_{[0,s]})\big) + \tau_{t+s}y(\cdot;u\cdot\chi_{[s,t+s]})\\
&= \tau_t x(s;\phi,u) + \tau_{t+s}y(\cdot;\tau_{-s}\tau_s(u\cdot\chi_{[0,t+s]}))\\
&= \tau_t x(s;\phi,u) + \tau_t y(\cdot;(\tau_s u)\cdot\chi_{[0,t]})\\
&= x(t;x(s,\phi,u),\tau_s u) \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&y(t+s;\phi,u)\\
&= \phi(t+s) + y(t+s;u)\\
&= \phi(t+s) + y(t+s;u\cdot\chi_{[0,s]}+\tau_{-s}\tau_s u)\\
&= \big(\tau_s\phi + \tau_s y(\cdot;u\cdot\chi_{[0,s]})\big)(t) + y(t;\tau_s u)\\
&= y(t;x(s;\phi,u),\tau_s u) \ . \quad \square
\end{aligned}
$$

The next result is strictly dual to Theorem 4.3 and we will only indicate the main idea of the proof.

THEOREM 4.4

Every ω - input-stable, time-invariant causal, linear input/output operator can be realized by a wellposed semigroup control system of the form (2.1).

LEMMA 4.2

Let T be any time invariant, causal, linear input-output operator. Then the following statements hold

(i) If $u \in W^{1,2}_{\ell oc}[\mathbb{R};U] \cap L^2_{0,\ell oc}[\mathbb{R};U]$ then $y = Tu \in W^{1,2}_{\ell oc}[\mathbb{R};U]$ and $\dot{y} = T\dot{u}$

(ii) If T is ω-input-stable and $\psi \in W^{1,2}_{\omega}[-\infty,0;U]$ with $\psi(0) = 0$, then $\phi = H\psi \in W^{1,2}_{\ell oc}[0,\infty;Y]$ with $\dot{\phi} = H\dot{\psi}$.

PROOF: The proof of statement (i) is strictly analogous to that of Lemma 3.2 and statement (ii) follows from density considerations as in the proof of Lemma 2.3. □

If T is the input-output operator associated with a wellposed semigroup control system of the form (2.1) via (2.5) then we say that T is realized by the control system (2.1), respectively by the operators A,B,C,T_μ.

THEOREM 4.3

Every ω-output-stable, time invariant, causal, linear input-output operator can be realized by a wellposed semigroup control system of the form (2.1).

PROOF: As our state space we choose $H = L^2_\omega[0,\infty;Y]$ and define

$$x(t;\phi,u) = \tau_t\phi + \tau_t y(\cdot;u\cdot\chi_{[0,t]}) , \tag{4.2}$$

$$y(t;\phi,u) = \phi(t) + y(t;u) \tag{4.3}$$

for $\phi \in H$, $u \in L^2_{\ell oc}[0,\infty;U]$, $t \geq 0$. Then equation (2.5) is an immediate consequence of (4.3). In view of Theorem 3.1 it therefore remains to show that the maps (4.2) and (4.3) define a time invariant, linear control system in the sense of section 3. The maps are obviously continuous and linear and satisfy the initial condition $x(0;\phi,u) = \phi$.

Now suppose that $u(t) = 0$ for $t < T$. Then $x(t;0,u) = \tau_t y(\cdot;0) = 0$ and $y(t;0,u) = y(t;u) = 0$ for $t < T$. This proves the causality and hence it remains to show that the system (4.2), (4.3) is time invariant. In fact, we obtain

associated with the extended input-output operator $\mathcal{T}$. This operator is obtained by extending the input $u \in L_0^2[-\infty,0;U]$ to all of $\mathbb{R}$ via $u(t) = 0$ for $t > 0$ and then restricting the associated output $y = \mathcal{T}u$ to the interval $[0,\infty)$. The Hankel operator associated with $\mathcal{T}^*$ is then given by

$$H^* : L_0^2[-\infty,0;Y] \longrightarrow L_{\ell oc}^2[0,\infty;U] \ .$$

Note that the Hankel operator, in general, does not give us the complete information about the input-output behavior of the system. In particular, the Hankel operator which corresponds to the control system (2.1) only depends on the operators A, B and C but not on T_μ. The simplest example of this situation is that $U = Y$ and $\mathcal{T}$ is the identity operator in which case H is zero. Nevertheless the Hankel operator plays a crucial role in realization theory. In particular, it serves for *characterizing the* stability behavior of the input-output operator $\mathcal{T}$.

<u>DEFINITION 4.1</u> (ω-stability)

<u>Let</u> $\omega \in \mathbb{R}$ <u>be given. Then the input-output operator</u> $\mathcal{T}$ <u>is said to be</u>

<u>(i)</u> <u>ω-output-stable, if</u> H <u>maps</u> $L_0^2[-\infty,0;U]$ <u>continuously into</u> $L_\omega^2[0,\infty;Y]$,

<u>(ii)</u> <u>ω-input-stable, if</u> H <u>extends to a continuous, linear operator from</u> $L_\omega^2[-\infty,0;U]$ <u>into</u> $L_{\ell oc}^2[0,\infty;Y]$,

<u>(iii)</u> <u>ω-stable if</u> H <u>extends to a bounded linear operator from</u> $L_\omega^2[-\infty,0;U]$ <u>into</u> $L_\omega^2[0,\infty;Y]$.

Of course, $\mathcal{T}$ is ω-input-stable if and only if $\mathcal{T}^*$ is ω-output-stable and vice versa. If $\mathcal{T}$ is the input-output operator associated with a well posed semigroup control system of the form (2.1) then it is ω-stable for every $\omega > \omega_0 = \lim_{t\to\infty} t^{-1}\log\|S(t)\|$ (Lemma 2.2 (iii)).

4. REALIZATION

Let the input space U and the output space Y both be Hilbert spaces. A continuous, linear input-output operator

$$\mathcal{T} : L^2_{0,\ell oc}[\mathbb{R};U] \longrightarrow L^2_{0,\ell oc}[\mathbb{R};Y]$$

is said to be <u>causal</u> if the implication

$$u(t) = 0 \ \forall\, t < T \implies \mathcal{T}u(t) = 0 \ \ \forall\, t < T$$

holds for all $T \in \mathbb{R}$. It is said to be <u>time invariant</u> if

$$\tau_t \mathcal{T} = \mathcal{T}\tau_t \tag{4.1}$$

for every $t \in \mathbb{R}$. Given such an operator $\mathcal{T}$ we sometimes use the notation $y(t;u) = \mathcal{T}u(t)$. Note that a linear input-output operator $\mathcal{T}$ is time invariant and causal if and only if its dual operator

$$\mathcal{T}^* : L^2_{0,\ell oc}[\mathbb{R};Y] \longrightarrow L^2_{0,\ell oc}[\mathbb{R};U]$$

has the same properties. At some places we refer to $\mathcal{T}$ as the extended input output operator. Via property (4.1) this extended operator is uniquely determined by its restriction to the interval $[0,\infty)$ which we still denote by

$$\mathcal{T} : L^2_{\ell oc}[0,\infty;U] \longrightarrow L^2_{\ell oc}[0,\infty;Y] .$$

Therefore every semigroup control system $\Sigma = (A,B,C,T_\mu)$ of the form (2.1) determines a time invariant, causal, linear input-output operator $\mathcal{T}$ via equation (2.5). The purpose of this section is to prove the converse statement.

To this end it is convenient to consider the <u>Hankel operator</u>

$$\mathcal{H} : L^2_0[-\infty,0;U] \longrightarrow L^2_{\ell oc}[0,\infty;Y]$$

$$x_0 - (\mu I-A)^{-1}Bu_0 = (\mu I-A)^{-1}\big(\mu x_0-Ax_0-Bu_0\big) \in W$$

with $u_0 = u(0)$. Then we obtain

$$\begin{aligned}
&y(0;x_0,u)\\
&= y(0;x_0-(\mu I-A)^{-1}Bu_0,0) + y(0;(\mu I-A)^{-1}Bu_0,u_0) + y(0;0,u-u_0)\\
&= C(\mu I-A)^{-1}\big(\mu x_0-Ax_0-Bu_0\big) + T_\mu u_0 \ .
\end{aligned}$$

This proves equation (2.4) for $t = 0$. In general (2.4) follows from the time invariance of the control system. $\square$

$$
\begin{aligned}
&\langle z_0, x(T;0,u) \rangle_H \\
&= \int_0^T \langle w(T-s;z_0), u(s) \rangle_U ds \\
&= \int_0^T \langle w(0;S^*(T-s)z_0), u(s) \rangle_U ds \\
&= \int_0^T \langle B^*S^*(T-s)z_0, u(s) \rangle_U ds \\
&= \langle z_0, \int_0^T S(T-s)Bu(s)ds \rangle_{V^*,V} .
\end{aligned}
$$

This shows that $x(t;x_0,u)$ is given by equation (2.3) for every $x_0 \in H$, every $u \in L^2_{\ell oc}[0,\infty;U]$ and every $t > 0$. In particular $x(\cdot;x_0,u) \in C^1[0,T;H]$ whenever $u \in W^{1,2}[0,T;U]$ and $Ax_0 + Bu(0) \in H$ (Lemma 2.2).

4. The operator family T_μ

For every $u_0 \in U$ we have the identity

$$A(\mu I-A)^{-1}Bu_0 + Bu_0 = \mu(\mu I-A)^{-1}Bu_0 \in H .$$

Hence it follows from Lemma 3.2 that $y(\cdot;(\mu I-A)^{-1}Bu_0,u_0) \in W^{1,2}[0,T;Y]$ where u_0 also denotes the constant function $u_0(t) \equiv u_0$. This allows us to define

$$T_\mu u_0 = y(0;(\mu I-A)^{-1}Bu_0,u_0) \tag{3.6}$$

for $u_0 \in U$ and $\mu \notin \sigma(A)$. The resolvent identity

$$(\mu I-A)^{-1} - (\lambda I-A)^{-1} = (\lambda-\mu)(\mu I-A)^{-1}(\lambda I-A)^{-1}$$

for $\mu,\lambda \in \sigma(A)$ implies the compatibility condition (2.2). It remains to establish equation (2.4) for all $x_0 \in H$ and $u \in W^{1,2}[0,T;U]$ with $Ax_0 + Bu(0) \in H$. For this purpose let us first note that in the case $u(0) = 0$ we get $y(0;0,u) = y(\varepsilon;0,\tau_{-\varepsilon}u) = 0$ since $y(\cdot;0,\tau_{-\varepsilon}u)$ is continuous (Lemma 3.2) and vanishes (almost) everywhere on the interval $[0,\varepsilon]$. In the general case we make use of the fact that

3. The Operator B

For every $z_0 \in H$ let us define $w(\cdot;z_0) \in L^2_{\ell oc}[0,\infty;U]$ by the identity

$$\int_0^T < w(T-s;z_0),u(s) >_U ds = < z_0,x(T;0,u) >_H \tag{3.3}$$

for every $T > 0$ and every $u \in L^2_{\ell oc}[0,\infty;U]$. It follows from the causality and time invariance that $w(t;z_0)$ is well defined. Moreover, the following equation holds for all $z_0 \in H$ and all $t,s > 0$

$$w(t+s;z_0) = w(t;S^*(s)z_0) \ . \tag{3.4}$$

This is a consequence of the identity

$$\begin{aligned}
\int_0^T < w(t;S^*(s)z_0),u(T-t) >_U dt &= < S^*(s)z_0,x(T;0,u) >_H \\
&= < z_0,S(s)x(T;0,u) >_H \\
&= < z_0, x(T+S;0,u\cdot\chi_{[0,T]}) >_H \\
&= \int_0^T < w(T+s-t;z_0),u(t) >_U dt \\
&= \int_0^T < w(t+s;z_0),u(T-t) >_U dt
\end{aligned}$$

for $u \in L^2_{\ell oc}[0,\infty;U]$. If $z_0 \in \mathcal{D}(A^*)$ then it follows from (3.4) in connection with Lemma 3.2 that $w(\cdot;z_0) \in W^{1,2}_{\ell oc}[0,\infty;U]$ and $\dot{w}(t;z_0) = w(t;A^*z_0)$ for almost every $t > 0$. This allows us to define $B \in \mathcal{L}(U,V)$ by

$$B^*z_0 = w(0;z_0), \quad z_0 \in V^* \tag{3.5}$$

Now let $u \in L^2_{\ell oc}[0,\infty;U]$ and $z_0 \in V^*$ be given. Then we obtain from (3.3), (3.4) and (3.5) that

LEMMA 3.2

Let $x_0 \in H$ *be the initial state and* $u \in W^{1,2}_{loc}[0,\infty;U]$ *be the input of a time invariant linear control system. Furthermore, suppose that* $x(t) = x(t;x_0,u)$ *is continuously differentiable in* H *for* $t \geq 0$. *Then* $y(\cdot;x_0,u) \in W^{1,2}_{loc}[0,\infty;Y]$ *and* $\dot{y}(t;x_0,u) = y(t;\dot{x}(0),\dot{u})$ *for almost every* $t \geq 0$.

PROOF: Note that $h^{-1}(\tau_h u-u)$ converges to $\dot{u}$ in $L^2_{loc}[0,\infty;U]$. Therefore the function

$$h^{-1}\big(y(t+h;x_0,u) - y(t;x_0,u)\big)$$

$$= y(t;h^{-1}(x(h;x_0,u)-x_0),h^{-1}(\tau_h u-u))$$

converges to $y(\cdot;\dot{x}(0),\dot{u})$ in $L^2[0,T;Y]$ for every $T > 0$ as h tends to zero. This proves the statement of the Lemma. □

PROOF OF THEOREM 3.1

1. The operator A

The operators $S(t) \in \mathcal{L}(H)$ defined by

(3.1) $$S(t)x_0 = x(t;x_0,0)$$

for $t \geq 0$, $x_0 \in H$, form a strongly continuous semigroup and we define A to be its infinitesimal generator. Furthermore, we introduce the Hilbert spaces $W = \mathcal{D}(A) \subset H$ and $V^* = \mathcal{D}(A^*) \subset H^*$ endowed with the respective graph norms so that $W \subset H \subset V$ with continuous dense injections. Then $S(t) \in \mathcal{L}(W) \cap \mathcal{L}(V)$ and $A \in \mathcal{L}(W,H) \cap \mathcal{L}(H,V)$.

2. The operator C

If $x_0 \in \mathcal{D}(A)$ then it follows from Lemma 3.2 with $u \equiv 0$ that $y(\cdot;x_0,0) \in W^{1,2}[0,T;Y]$ with $\dot{y}(t;x_0,0) = y(t;Ax_0,0)$. In particular, $y(t;x_0,0)$ is continuous and we can define $C \in \mathcal{L}(W,Y)$ by

(3.2) $$Cx_0 = y(0;x_0,0), \quad x_0 \in W .$$

6. CONTINUITY AND BOUNDEDNESS

For the semigroup control system $\Sigma = (A,B,C,T_\mu)$ the smoothness of the output is related to the degree of boundedness of the output operator C in the following way.

REMARK 6.1

Let $\Sigma = (A,B,C,T_\mu)$ be a well posed semigroup control system and let $\mathcal{C} : H \to L^2_{\ell oc}[0,\infty;Y]$ be defined by $\mathcal{C}x_0 = CS(t)x_0$ for $x_0 \in W$ and $t > 0$. Then

(i) $C \in \mathcal{L}(H,Y)$ if and only if $\mathcal{C}$ maps H continuously into $C_{\ell oc}[0,\infty;Y]$,

(ii) $C \in \mathcal{L}(H,Y)$ satisfies an inequality of the form

$$\int_0^T \|CS(t)x_0\|_Y^2 dt \le c^2\|x_0\|_V^2 , \; x_0 \in H , \tag{6.1}$$

for some constants $T > 0$, $c > 0$ if and only if $\mathcal{C}$ maps H continuously into $W^{1,2}_{\ell oc}[0,\infty;Y]$,

(iii) $C \in \mathcal{L}(V,Y)$ if and only if $\mathcal{C}$ maps H continuously into $C^1_{\ell oc}[0,\infty;Y]$.

Note that the assertions (ii) and (iii) follow from the fact that $d/dt CS(t)x_0 = CS(t)Ax_0$ and that $\mu I-A:H \to V$ is an isomorphism for $\mu \notin \sigma(A)$.

Conversely, the smoothing properties of the Hankel operator imply the existence of a realization with a "well behaved" output operator C.

PROPOSITION 6.2

Let $\mathcal{T}$ be an ω-input-stable, time invariant, causal, linear input-output operator and let $\mathcal{H} : L^2_\omega[-\infty,0;U] \to L^2_{\ell oc}[0,\infty;Y]$ be the associated extended Hankel operator. Then the following statements hold.

(i) If $\mathcal{H}$ maps $L^2_\omega[-\infty,0;U]$ continuously into $C_{\ell oc}[0,\infty;Y]$ then there exists a realization $\Sigma = (A,B,C,T_\mu)$ of $\mathcal{T}$ such that

$$\omega = \lim_{t\to\infty} t^{-1} \log\|S(t)\| \tag{6.2}$$

and $C \in \mathcal{L}(H,Y)$.

(ii) If H *maps* $L^2_\omega[-\infty,0;U]$ *continuously into* $W^{1,2}_{\ell oc}[0,\infty;Y]$ *then there exists a realization* $\Sigma = (A,B,C,T_\mu)$ *of* T *such that* (6.2) *holds and* $C \in L(H,Y)$ *satisfies* (6.1) *for some constants* $T > 0$, $c > 0$.

(iii) If H *maps* $L^2_\omega[-\infty,0;U]$ *continuously into* $C^1_{\ell oc}[0,\infty;Y]$ *then there exists a realization* $\Sigma = (A,B,C,T_\mu)$ *of* T *such that* (6.2) *holds and* $C \in L(V,Y)$.

PROOF: Consider the realization Σ_U of T given by (4.7). Then

$$C_U S_U(t)\psi = H\psi(t)$$

for $\psi \in H_U = L^2_\omega[-\infty,0;U]$. Hence the statements of the proposition follow from Remark 6.1. □

By duality, the properties of the input operator B in the realization are related to the smoothness of the dual Hankel operator H^*.

PROPOSITON 6.3

Let T *be a time invariant, causal, linear input-output operator with the associated Hankel operator* H *and let* $\omega_1 \in R$ *be given. Then the following statements hold.*

(i) H^* *extends to a continuous map from* $L^2_\omega[-\infty,0;Y]$ *into* $C_{\ell oc}[0,\infty;U]$ *for some* $\omega < \omega_1$ *if and only if there exists a realization* $\Sigma = (A,B,C,T_\mu)$ *of* T *such that*

$$\lim_{t\to\infty} t^{-1}\log\|S(t)\| < \omega_1 \tag{6.3}$$

and $B \in L(U,H)$.

(ii) H^* *extends to a continuous map from* $L^2_\omega[-\infty,0;Y]$ *into* $W^{1,2}_{\ell oc}[0,\infty;Y]$ *for some* $\omega < \omega_1$ *if and only if there exists a realization* $\Sigma = (A,B,C,T_\mu)$ *of* T *such that* (6.3) *holds and* $B \in L(U,H)$ *satisfies*

$$\left\|\int_0^T S(T-s)Bu(s)ds\right\|_W < \|u\|_{L^2[0,T;U]} \tag{6.4}$$

for some constants $T > 0$, $c > 0$.

(iii) $\mathcal{H}^*$ extends to a continuous map from $L^2_\omega[-\infty,0;Y]$ into $C^1_{\ell oc}[0,\infty;U]$ for some $\omega < \omega_1$ if and only if there exists a realization $\Sigma = (A,B,C,T_\mu)$ of $\mathcal{T}$ such that (6.3) holds and $B \in \mathcal{L}(U,W)$.

Thus we have characterized those input-output operators $\mathcal{T}$ which can be realized by a state space system in which either the input operator B or the output operator C is bounded. It seems to be a much more delicate problem to find necessary and sufficient conditions under which there exists a realization $\Sigma = (A,B,C,T_\mu)$ with both $B \in \mathcal{L}(U,H)$ and $C \in \mathcal{L}(H,Y)$. A necessary condition is of course that $\mathcal{T}$ can be represented in the form

$$\mathcal{T}u(t) = Du(t) + \int_{-\infty}^{t} G(t-s)u(s)ds \tag{6.5}$$

where $D \in \mathcal{L}(U,Y)$ and $G(t) \in \mathcal{L}(U,Y)$, $t > 0$, is strongly continuous and satisfies some exponential bound. A sufficient condition is contained in Proposition 6.2 (iii) and Proposition 6.3 (iii). Somewhere inbetween lies the following result.

THEOREM 6.4

Let $\mathcal{T}$ be a time invariant, causal, linear input-output operator with the associated Hankel operator $\mathcal{H}$. Let $\omega_1 \in \mathbb{R}$ be given. Then the following statements are equivalent.

(i) $\mathcal{H} \in \mathcal{L}(L^2_\omega[-\infty,0;U], W^{1,2}_\omega[0,\infty;Y])$ for some $\omega < \omega_1$.

(ii) $\mathcal{H}^* \in \mathcal{L}(L^2_\omega[-\infty,0;Y], W^{1,2}_\omega[0,\infty;U])$ for some $\omega < \omega_1$.

(iii) There exist Hilbert spaces $Z \subset X$ with a continuous, dense injection and operators $A \in \mathcal{L}(Z,X)$, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(Z,Y)$, $D \in \mathcal{L}(U,Y)$ such that A is the infinitesimal generator of a strongly continuous semigroup $S(t) \in \mathcal{L}(X) \cap \mathcal{L}(Z)$ with $\lim_{t\to\infty} t^{-1}\log\|S(t)\| < \omega_1$. Furthermore, the inequalities

$$\left\|\int_0^T S(T-s)Bu(s)ds\right\|_Z \le c\|u\|_{L^2[0,T;U]}, \tag{6.6}$$

$$\|CS(\cdot)x_0\|_{L^2[0,T;Y]} \le c\|x_0\|_X, \quad x_0 \in Z, \tag{6.7}$$

hold for some constants $T > 0$, $c > 0$ and the operator $\mathcal{T}$ is given by

$$\mathcal{T}u(t) = Du(t) + C\int_{-\infty}^{t} S(t-s)Bu(s)ds,\ t \in \mathbb{R}, \tag{6.8}$$

for $u \in L^2_{0,\ell oc}[\mathbb{R},U]$.

PROOF: Proposition 6.3 (ii) shows that statement (ii) holds if and only if there exists a realization $\Sigma = (A,B,C,T_\mu)$, with (6.3) and $B \in L(U,H)$ satisfying (6.4). But this is exactly equivalent to statement (iii) with $X = H$ and $Z = W$. In particular, it follows from (2.2) that the operator

$$D = T_\mu - C(\mu I-A)^{-1}B \in L(U,Y)$$

is independent of μ and therefore $\mathcal{T}$ is given by (6.8). The equivalence of the statements (i) and (iii) follows by duality. $\square$

In the remainder of this section we briefly discuss the case that the input space $U = \mathbb{R}^m$ and the output space $Y = \mathbb{R}^p$ are finite dimensional.

Note that the second statement of the following theorem is a modified version of a result by YAMAMOTO [26].

THEOREM 6.5

Suppose that $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$ and let $\mathcal{T} : L^2_{0,\ell oc}[\mathbb{R};U] \to L^2_{0,\ell oc}[\mathbb{R};Y]$ be a time invariant, causal, linear input-output operator. Then the following statements hold.

(i) $\mathcal{T}$ admits a realization $\Sigma = (A,B,C,T_\mu)$ with either $C \in L(H,Y)$ or $B \in L(U,H)$ iff $\mathcal{T}$ is given by (6.5) with $D \in \mathbb{R}^{p\times m}$ and $G \in L^2_\omega[0,\infty;\mathbb{R}^{p\times m}]$ for some $\omega \in \mathbb{R}$.

(ii) $\mathcal{T}$ admits a realization $\Sigma = (A,B,C,T_\mu)$ with either $C \in L(V,Y)$ or $B \in L(U,W)$ iff $\mathcal{T}$ is given by (6.5) with $D \in \mathbb{R}^{p\times m}$ and $G \in W^{1,2}_\omega[0,\infty,\mathbb{R}^{p\times m}]$ for some $\omega \in \mathbb{R}$.

PROOF: If T is given by (6.5) with $G \in L^2_\omega[0,\infty;R^{p\times m}]$ then H is given by

$$H\psi(t) = \int_{-\infty}^{0} G(t-s)\psi(s)ds\ , \quad t > 0\ , \tag{6.9}$$

and therefore maps $L^2_\omega[-\infty,0;R^m]$ continuously into $C_{\ell oc}[0,\infty;R^p]$. Conversely suppose that H maps $L^2_\omega[-\infty,0;R^m]$ continuously into $C_{\ell oc}[0,\infty;R^p]$. Then it follows from the Riesz representation theorem that there exists a function $G \in L^2_\omega[0,\infty;R^{p\times m}]$ such that

$$H\psi(0) = \int_{-\infty}^{0} G(-s)\psi(s)ds$$

for all $\psi \in L^2_\omega[-\infty,0;R^m]$. Using the time invariance we obtain that H is given by (6.9). Now define the input-output operator $\tilde{T}$ by (6.5) with $D = 0$. Then the equations (4.7) show that T and $\tilde{T}$ admit the same realization except for a possible difference in the operators T_μ and $\tilde{T}_\mu$. Hence it follows from (2.2) that the matrix $T_\mu - \tilde{T}_\mu = D \in R^{p\times m}$ is independent of μ and therefore T is given by (6.5). This proves statement (i).

If T is given by (6.5) with $G \in W^{1,2}_\omega[0,\infty;R^{p\times m}]$ then it follows from (6.9) that $H\psi \in C^1_{\ell oc}[0,\infty;R^p]$ for every $\psi \in L^2_\omega[-\infty,0;R^m]$ and

$$d/dt\ H\psi(t) = \int_{-\infty}^{0} \dot{G}(t-s)\psi(s)ds\ .$$

Hence T admits a realization $\Sigma = (A,B,C,T_\mu)$ with $C \in L(V,Y)$ (Proposition 6.2 (ii)). Conversely, suppose that H maps $L^2_\omega[-\infty,0;R^m]$ continuously into $C^1_{\ell oc}[0,\infty;R^p]$. Then T is given by (6.5) for some matrix $D \in R^{p\times m}$ and some function $G \in L^2_\omega[0,\infty;R^{p\times m}]$. Furthermore, there exists a function $K \in L^2_\omega[0,\infty;R^{p\times m}]$ such that

$$d/dt\, H\psi(t) = \int_{-\infty}^{0} K(t-s)\psi(s)ds\ , \quad t > 0\ ,$$

for all $\psi \in L^2_\omega[-\infty,0;R^m]$. This follows again from the Riesz representation theorem and from the time invariance. We conclude that

$$\int_{-\infty}^{0} \Big(G(T-s) - G(-s) - \int_0^T K(t-s)dt\Big)\psi(s)ds$$

$$= H\psi(T) - H\psi(0) - \int_0^T d/dt\, H\psi(t)dt = 0$$

for all $\psi \in L_0^2[-\infty,0;R^m]$ and hence G is absolutely continuous with $\dot{G} = K$. This proves statement (ii). $\square$

It is a much more difficult problem to characterize the requirements of Theorem 6.4 in terms of the function $G(t)$. It follows from Theorem 6.5 that the operator T can be written in the form (6.5) with $G \in L_\omega^2[0,\infty;R^{p\times m}]$ if the statements of Theorem 6.4 are satisfied. Furthermore, a sufficient condition is that $G(t) \in R^{p\times m}$ is locally of bounded variation and satisfies an estimate of the form

$$\underset{[0,t]}{\mathrm{VAR}}\, G \leq Me^{\omega t} \tag{6.10}$$

for some $\omega < \omega_1$, $M > 1$. In this case we have

$$d/dt \quad H\psi(t) = \int_t^\infty dG(s)\psi(t-s) \tag{6.11}$$

for almost every $t > 0$ and every $\psi \in L_\omega^2[-\infty,0;R^m]$. But G need not be of bounded variation. In the same way there exist time invariant, causal, linear input-output operators T which cannot be represented in the form

$$Tu(t) = \int_0^\infty d\mu(s)u(t-s) \;, \qquad t \in R \;, \tag{6.12}$$

for some matrix function $\mu(t)$ of bounded variation.

7. CONCLUSIONS

We have established a quite general realization theorem for infinite dimensional, linear control systems using a state space approach with unbounded input and output operators. The whole theory can be developed with minor alterations if the assumption on ω-stability is dropped and the state space is allowed to be a Fréchet space rather than a Hilbert space. For single-input/single-output systems and bounded input and output operators this has been done by YAMAMOTO [26]. Example 5.1, however, shows that such an approach might not always be advantageous.

An interesting and nontrivial problem might be to characterize those input-output operators T which can be realized by a state space model $\Sigma = (A,B,C,T_\mu)$ where both the input operator $B \in L(U,H)$ and the output operator $C \in L(H,Y)$ are bounded. Results in this direction have been developed by AUBIN-BENSOUSSAN [1], BENSOUSSAN-DELFOUR-MITTER [4], FUHRMANN [10].

We also mention the problem of characterizing those operator families T_μ which represent time invariant, linear input-output operators T in the frequency domain. This problem has been studied by PANDOLFI [17], ROUCHALEAU-SONTAG [20], FUHRMANN [10] and others.

Another nontrivial problem seems to be under which conditions the ω-stability of the input-output operator implies the ω-stability of the semigroup in the realization.

Finally there is the question of how to deal with non-wellposed input-output operators.

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REFERENCES

[1] J. P. AUBIN/A. BENSOUSSAN, Models of representation of linear invariant systems in continuous time, MRC, University of Wisconsin, TSR #1286, 1973.

[2] J. S. BARAS/R. W. BROCKETT, H^2-functions and infinite dimensional realization theory, SIAM J. Control 13 (1975), 221-224.

[3] J. S. BARAS/R. W. BROCKETT/P. A. FUHRMANN, State space models for infinite dimensional systems, IEEE Trans. Autom. Control AC-19 (1974), 693-700.

[4] A. BENSOUSSAN/M. C. DELFOUR/S. K. MITTER, Representation theory for linear infinite dimensional continuous time systems, in "Mathematical Systems Theory", G. Marchesini & S. K. Mitter, eds., pp. 204-225, LNEMS 131, Springer Vertag, New York 1976.

[5] R. F. CURTAIN, Sufficient conditions for infinite-rank Hankel operators to be nuclear, Mathematics Institute, University of Groningen, 1984.

[6] M. C. DELFOUR/A. MANITIUS, The structural operator F and its role in the theory of retarded systems, J. Math. Anal. Appl. 73 (1980), 466-490, J. Math. Anal. Appl. 74 (1980), 359-381.

[7] R. EISING/M. L. J. HAUTUS, Realization algorithms for systems over a principle ideal domain, Math. Systems Theory 14 (1981), 353-366.

[8] P. A. FUHRMANN, On realization of linear systems and applications to some questions of stability, Math. Systems Theory 8 (1974), 132-141.

[9] P. A. FUHRMANN, Algebraic system theory: an analyst's point of view, J. Franklin Inst. 301 (1976), 521-540.

[10] P. A. FUHRMANN, Linear Systems and Operators in Hilbert Space, McGraw Hill, New York, 1981.

[11] R. E. KALMAN, On partial realization, transfer functions and canonical forms, Acta Polytechnica Scandinavia 31 (1979), 9-32.

[12] R. E. KALMAN/M. ARBIB/P. FALB, Topics in Mathematical System Theory, McGraw-Hill, New York, 1969.

[13] R. E. KALMAN/M. L. J. HAUTUS, Realization of continuous-time linear dynamical systems: Rigorous theory in the style of Schwartz, in "Ordinary Differential Equations", L. Weiss, ed., pp. 151-164, Academic Press, New York, 1972.

[14] E. W. Kamen, Representation of linear continuous-time systems by spaces of distributions, IRIA report INF 7213/72016, June, 1972.

[15] E. W. KAMEN, On an algebraic theory of systems defined by convolution operators, Math. Systems Theory 9 (1975), 57-74.

[16] E. W. KAMEN, An operator theory for linear functional differential equations, J. Diff. Equations 27 (1978), 274-297.

[17] L. PANDOLFI, Canonical realization of systems with delays, SIAM J. Control Opt. 21 (1983), 598-613.

[18] A. PAZY, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Vertag, New York, 1984.

[19] A. J. PRITCHARD/D. SALAMON, The linear quadratic control problem for infinite dimensional systems, Part I: A semigroup theoretic approach for systems with unbounded input and output operators, Part II: Retarded systems with delays in control and observation, MRC, University of Wisconsin, TSR #2624, 1984.

[20] Y. ROUCHALEAU/E. D. SONTAG, On the existence of minimal realizations of linear dynamical systems over Noetherian integral domains, J. Computer Syst. Science 18 (1975), 65-75.

[21] D. SALAMON, Control and Observation of Neutral Systems, RNM 91, Pitman, London, 1984.

[22] D. SALAMON, Infinite dimensional systems with unbounded control and observation: a functional analytic approach, MRC, University of Wisconsin, TSR #2794, 1985.

[23] E. D. SONTAG, Linear systems over commutative rings: a survey, Ricerche di Automatica 7 (1977), 1-34.

[24] E. D. SONTAG, On split realization of response maps over rings, Inform. and Control 37 (1978), 23-33.

[25] M. W. SPONG, Hereditary realizations of linear neutral delay-differential systems, School of Electrical Engineering, Cornell University, 1984.

[26] Y. YAMAMOTO, Realization theory of infinite-dimensional lienar systems, Part I & II, Math. Systems Theory 15 (1981/82), 55-77, 169-190.

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This paper studies the state space representation of time invariant causal, linear input-output operators in continuous time. A realization theorem in the Hilbert space context is established in full generality using unbounded input and output operators. The first step is to construct an abstract state space representation with a prescribed input-output behavior. The second step is to transform this abstract system into a differential equation. The uniqueness problem is discussed in some detail as well as the relation to existing results in realization theory.

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